JOURNAL OF COMBINATORIAL THEORY, Series B 27, 42–48 (1979)

Brooks' Graph-Coloring Theorem and the Independence Number*

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Communicated by C. St. J. A. Nash-Williams

Received August 4, 1976

Let h denote the maximum degree of a connected graph H, and let $\chi(H)$ denote its chromatic number. Brooks' Theorem asserts that if $h \geqslant 3$, then $\chi(H) \leqslant h$, unless H is the complete graph K_{h+1}. We show that when H is not K_{h+1}, there is an h-coloring of H in which a maximum independent set is monochromatic. We characterize those graphs H having an h-coloring in which some color class consists of vertices of degree h in H. Again, without any loss of generality, this color class may be assumed to be maximum with respect to the condition that its vertices have degree h.

1. NOTATION

We shall follow the notation of Harary [3]. All graphs in this paper are simple. Let $V(G)$ denote the vertex set of the graph G, and let $E(G)$ denote the edge set. We shall assume that $E(G)$ is nonempty. Thus, the maximum degree $\Delta(G)$ of the vertices of G is at least 1.

For any set X, we let $|X|$ denote the cardinality of X. To simplify notation, we denote the singleton set $\{x\}$ by x, so that the union of a set S and that singleton may be written $S + x$.

A *coloring* of G is a partition of $V(G)$ into independent subsets, where the partition is unordered and admits null sets. A set $X \subseteq V(G)$ is *monochromatic* in a coloring of G if all vertices of X have the same color. The *chromatic number* $\chi(G)$ of G is the minimum possible number of sets in a coloring of G.

A θ-graph is a graph consisting of three distinct arcs, joining the same two vertices and having no other common vertices.

2. INTRODUCTION

The basic result in the literature on the problem of coloring a graph G of specified maximum degree is Brooks' Theorem [2]:

* This paper is part of the author's Ph.D. thesis, done at the Ohio State University under Professor Neil Robertson.

0095-8956/79/040042-07$2.00/0

THEOREM 2.1. *Let G be a graph with maximum degree $\Delta(G)$. We have*

$$\chi(G) \leqslant \Delta(G) + 1. \tag{2.1}$$

If $\Delta(G) = 2$, then equality holds in (2.1) *if and only if G contains an odd cycle. If $\Delta(G) \neq 2$, then equality holds if and only if G contains a clique $K_{\Delta(G)+1}$.*

Note that if $\Delta(G) = 2$, an odd cycle of G is necessarily a connected component of G. Also, a clique $K_{\Delta(G)+1}$ is necessarily a component of G. Such components, which force equality in (2.1), are called *$B_{\Delta(G)}$-components.*

Since each component of a graph can be colored independently, we can assume without loss of generality, that G is connected.

We give a proof of Brooks' Theorem by induction on $\Delta(G)$, and in so doing, we obtain new information. For instance, we show that if G is not a $B_{\Delta(G)}$-component, then there is a coloring of G in $\Delta(G)$ colors in which some monochromatic set has $\beta(G)$ vertices. Also, we characterize those connected graphs G for which there is a coloring of G in $\Delta(G)$ colors such that some monochromatic set consists solely of vertices of degree $\Delta(G)$.

Albertson, Bollobás, and Tucker [1] showed first that with two exceptions H_1 and H_2 defined below, every graph H with $\Delta(H) = h$ and with no subgraph K_h has independence number

$$\beta(H) > |V(H)|/h,$$

and they conjectured that such graphs H have an h-coloring in which some monochromatic set has more than $|V(H)|/h$ vertices. Second, they proved this conjecture for graphs that are not regular of degree h. Theorem 3.2, below, combined with the first result of Albertson, Bollobás, and Tucker shows that this conjecture is true, even for regular graphs.

The two exceptional graphs H_1 and H_2 may be defined as follows: let $V(H_1)$ be the integers modulo 8, and let $\{v, w\} \in E(H_1)$ if and only if

$$v - w \equiv 1, 2, 6, \text{ or } 7 \pmod{8}.$$

Let $V(H_2)$ be the integers modulo 10, and let $\{v, w\} \in E(H_2)$ if and only if

$$v - w \equiv 1, 4, 5, 6, \text{ or } 9 \pmod{10}.$$

3. THE MAIN RESULTS

In this section, we shall consider a connected graph H with at least one edge. To simplify notation, we denote $\Delta(H)$ by h.

A largest independent subset of the set of vertices of degree h in H will be called a *superstable set.*

The equivalence of (3.4) and (3.6) of Theorem 3.2 below is Brooks' Theorem (Theorem 2.1).

A *Brooks tree* is any graph H with $\Delta(H) = h$ that arises from a tree T satisfying $\Delta(T) \leqslant h$ by the replacement of each vertex of T with

(a) an odd cycle if $h = 3$;

(b) a clique K_h if $h \neq 3$,

such that if x and y are adjacent vertices of T, then the cycles or cliques substituted for x and y are joined by an edge whose removal disconnects H. Thus, K_2 is the only Brooks tree with $h = 1$; odd arcs with at least 3 edges are the only Brooks trees with $h = 2$; and if $h \geqslant 3$, then a Brooks tree is not a tree.

THEOREM 3.1. *Let H be a connected graph with $\Delta(H) = h \geqslant 1$. The following are equivalent*:

(3.1) *H is a B_h-component, or a Brooks tree*;

(3.2) *There is no superstable set S such that $H - S$ can be colored in $h - 1$ colors*;

(3.3) *There is no independent set S of vertices of degree h such that $H - S$ can be colored in $h - 1$ colors.*

We also have

THEOREM 3.2. *Let H be a connected graph with $\Delta(H) = h \geqslant 1$. The following are equivalent*:

(3.4) *H is a B_h-component*;

(3.5) *There is no maximum independent set S, such that $H - S$ can be colored in $h - 1$ colors*;

(3.6) *There is no h-coloring of H.*

Proof of Theorem 3.2 *from Theorem* 3.1. For $\Delta(H) \leqslant 2$, the theorem is easily verified. Assume therefore, that $\Delta(H) \geqslant 3$.

We show that if (3.1), (3.2), and (3.3) are equivalent for $\Delta(H) = h$, then (3.4), (3.5), and (3.6) are also equivalent for $\Delta(H) = h$. Since (3.4) implies (3.6) and (3.6) implies (3.5), it suffices to prove that (3.5) implies (3.4) if (3.1), (3.2), and (3.3) are equivalent.

Adjoin to H a set V of $\sum (h - \deg_H(v))$ vertices disjoint from $V(H)$, where the sum runs over all $v \in V(H)$. We join each vertex v of H to exactly $h - \deg_H(v)$ vertices of V, such that no vertex of V is joined to more than one vertex of H. Denote the resulting graph H'. Then, (3.7) $H'[V(H)] = H$;

(3.8) Any $v \in V(H)$ has degree h in H';

(3.9) Any $v \in V$ has degree 1 in H'.

By (3.7) and (3.8), a superstable set S in H' is a maximum independent set in H. Hence, (3.5) for H implies (3.2) for H', whence by (3.1), either H' is a B_h-component, or it is a Brooks tree. Since Brooks trees have vertices of degree $h - 1$, conditions (3.8), (3.9), and $h \geqslant 3$ imply that H' is not a Brooks tree. Thus, H' is a B_h-component, and, therefore, has no vertices of degree 1, whence $H = H'$. This proves (3.4), and thus the equivalence of (3.4), (3.5), and (3.6). Hence, Theorem 3.2 follows from Theorem 3.1.

Proof of Theorem 3.1. Again, we may suppose that $h \geqslant 3$. Since (3.1) implies (3.3) and (3.3) implies (3.2), it suffices to show that (3.2) implies (3.1).

Suppose inductively that the theorem is true for all graphs G with $\Delta(G) < h$. Then Theorem 3.2 is true for such graphs G. Let H be a graph with $\Delta(H) = h$ such that H does not satisfy (3.1), and such that for any superstable set S, $H - S$ has no $(h - 1)$-coloring. For a given superstable set S, Theorem 3.2 and

$$\Delta(H - S) \leqslant h - 1$$

imply that either $H - S$ can be colored in $h - 1$ colors, or $H - S$ has a B_{h-1}-component. We have already precluded the first possibility. Hence, $H - S$ has a B_{h-1}-component. Without loss of generality, we shall choose S to be a superstable set that minimizes the number of B_{h-1}-components in $H - S$.

Suppose that a vertex $s \in V(H)$ is in no B_{h-1}-component in $H - S$, regardless of the choice of a superstable set S that minimizes the number of B_{h-1}-components in $H - S$. Since H is connected, such a vertex s exists that is adjacent to a vertex v lying in a B_{h-1}-component C of $H - S$, for some such S. Since the only vertex not in C that is adjacent to v lies in S, we must have $s \in S$. Then $S + v - s$ is a superstable set, and either $H - (S + v - s)$ has one fewer B_{h-1}-component than $H - S$, contrary to the choice of S, or s lies in a B_{h-1}-component of $H - (S + v - s)$, contrary to the choice of s. Hence, by contradiction, all vertices of H lie in B_{h-1}-components of $H - S$, for suitable S.

Case I. Suppose that there is no cycle P in H with the property that it is not contained in a B_{h-1}-component of $H - S$, regardless of the choice of superstable set S that minimizes the number of B_{h-1}-components in $H - S$. Observe that if $h = 3$, then any even cycle has this property, and any cycle not contained in a clique has this property if $h > 3$. Hence, since P does not exist,

$$\text{If } P' \text{ is a cycle in } H \text{ and if } h = 3, \text{ then } P' \text{ has odd girth} \qquad (3.10)$$

and

If P' is a cycle in H and $h \geqslant 4$, then P' is contained in a clique. (3.11)

We see that if H is a Brooks tree or a B_h-component, then (3.10) holds if $h = 3$, and (3.11) holds if $h > 3$. However, we shall prove the converse: i.e., if either (3.10) or (3.11) holds, then H is either a Brooks tree or a B_h-component. The main step in proving this converse lies in showing that

(3.12) If H is not K_{h+1}, then H can be partitioned into:

(i) odd cycles, if $h = 3$ and (3.10) holds or

(ii) cliques K_h if $h > 3$ and (3.11) holds.

Then it follows from (3.10) or (3.11), from (3.12), and from the definition of Brooks trees that H must be either K_{h+1} or a Brooks tree.

Thus, if P does not exist as described, then (3.10) or (3.11) holds, depending upon the value of h, and (3.1) follows (which is what we were to prove), provided (3.12) is proven. We shall, therefore, complete Case I by proving (3.12).

We have already shown that each vertex of H lies in a B_{h-1}-component of $H - S$ for some superstable set S that minimizes the number of B_{h-1}-components in $H - S$. If we show that each vertex of H lies in exactly one such B_{h-1}-component, then (3.12) follows.

Suppose, therefore, that some vertex of H lies in a B_{h-1}-component C_1 of $H - S_1$ and in a B_{h-1}-component C_2 of $H - S_2$, for suitable S_1 and S_2, where C_1 and C_2 are distinct. We shall now derive contradictions with (3.10) and (3.11). If $h \geqslant 4$, then C_1 and C_2 are cliques on h vertices each. Since they overlap, $\Delta(H) = h$ forces

$$|V(C_1) \cup V(C_2)| \leqslant h + 1,$$

and since C_1 and C_2 are distinct, we have equality. Hence, $H[V(C_1) \cup V(C_2)]$ is either isomorphic to K_{h+1} or to K_{h+1} minus an edge. In the first case, H is K_{h+1}. In the second case, the cycle P' of 4 vertices in $H[V(C_1) \cup V(C_2)]$ containing the two nonadjacent vertices violates (3.11). On the other hand, if $h = 3$, then C_1 and C_2 are overlapping odd cycles, and $\Delta(H) = h < 4$ forces them to overlap in an edge. Then $C_1 \cup C_2$ contains a θ-graph, and hence an even cycle, contrary to (3.10).

Case II. Suppose that there is a cycle P in H that is not contained in a B_{h-1}-component of $H - S$, where S is superstable and the number of B_{h-1}-components of $H - S$ is minimized. This number of B_{h-1}-components of $H - S$ is positive, for otherwise, since the induction hypothesis on h implies that (3.4) and (3.6) are equivalent for $H - S$, we would have $\chi(H - S) \leqslant h - 1$, contrary to (3.2). Since every vertex of H lies in a B_{h-1}-component of

$H - S$, for suitable S, we can choose $S = S_0$ so that a B_{h-1}-component C_0 of $H - S_0$ contains a vertex of P. Furthermore, we can assume without loss of generality that $| V(P) \cap S_0 |$ is minimized, with respect to these conditions.

Since the degree of any vertex of C_0 in $H - S_0$ is $h - 1$, and since $\Delta(H) = h$, an edge of P lies in $E(C_0)$. Since P is not contained in C_0, which is an induced subgraph of H, an edge of P lies outside $E(C_0)$. Therefore, there is a vertex v_1 of $V(P) \cap V(C_0)$ having one incident edge of P in $E(C_0)$ and the other incident edge of P, say $\{v_1, s_1\}$, outside $E(C_0)$. Since C_0 is a component in $H - S_0$, we have $s_1 \in S_0$.

We define the *path determined by* P to be the closed path consisting of successive vertices of the cycle P, where the first and second vertices in the path are v_1 and s_1, respectively. Denote the vertices of $P \cap S_0$ by $s_1, s_2, \ldots, s_m$, so that as one travels along the path determined by P, one encounters the vertices of $V(P) \cap S_0$ in the order $s_1, s_2, \ldots, s_m$. For each $i \leqslant m$, let v_i be the vertex preceding s_i in the path determined by P. Define a sequence $S_1, S_2, \ldots, S_{m-1}$ of sets inductively by

$$S_i = S_{i-1} + v_i - s_i .$$

Since S_0 is superstable, so is S_1. Since S_0 was chosen to minimize the number of B_{h-1}-components in $H - S_0$, and since C_0 is a B_{h-1}-component of $H - S_0$ but not of $H - S_1$, it follows that s_1 must be contained in a B_{h-1}-component C_1 of $H - S_1$, and S_1 also minimizes the number of B_{h-1}-components in $H - S_1$. Note that C_1 must contain v_2, and v_2 is adjacent to exactly one vertex of S_1, namely, s_2. By repeating this, one sees inductively that for $i = 1, 2, \ldots, m - 1$, S_i is a superstable set such that $H - S_i$ has the minimum possible number of B_{h-1}-components, and in particular, $H - S_i$ has a B_{h-1}-component C_i containing the vertices s_i and v_{i+1} of P. By definition of s_m and of the path determined by P, s_m is adjacent to a vertex of $V(C_0)$. By the minimality of $| V(P) \cap S_0 |$, which we assumed without loss of generality, s_m is the first vertex after s_1 along the path to be adjacent to a vertex of $V(C_0)$. Otherwise, if s_n were the first and $n < m$, then P could have been formed by passing from s_n through C_0 to v_1 directly, instead of prodeeding from s_n to C_0 and v_1 by way of s_{n+1}. This would violate the minimality of $| V(P) \cap S_0 |$. Since s_m is the first vertex after s_1 along the path determined by P to be adjacent to any vertex of C_0, the component $C_0 - v_1$ of $H - S_1$ is also a component of $H - S_{m-1}$. Hence, we can define

$$S_m = S_{m-1} + v_m - s_m ,$$

knowing not only that S_m is superstable and that the number of B_{h-1}-components of $H - S_m$ is minimized, but also that s_m is contained in a

B_{h-1}-component C_m of $H - S_m$ whose vertices are precisely s_m and $V(C_0) - v_1$. It must follow from the structure of B_{h-1}-components that

$$N(s_m) - v_m = N(v_1) - s_1,$$

where $N(v)$ denotes the set of vertices of H adjacent to v.

If C_0 is a cycle of girth at least 5, then s_m is adjacent to two nonadjacent vertices x_1, x_2 of degree $h = 3$ that comprise $N(v) - s$. Since s_m is the only vertex in S_0 to which x_1 and x_2 are adjacent, $S_0 \cup \{x_1, x_2\} - s_m$ is a bigger superstable set than S_0, contrary to the maximality of S_0.

If C_0 is a clique K_h, $h \geqslant 3$, then s_m is adjacent to every vertex of $C_0 - v_1$. If v_1 and s_m are adjacent, then $m = 1$, and $V(C_0) + s_m$ induces a clique K_{h+1} in H. Since H is connected, K_{h+1} is necessarily all of H, a case excluded since (3.1) is false. Suppose, therefore, that s_m and v_1 are not adjacent. Let x be a member of the equal sets $V(C_m - s_m) = V(C_0 - v_1)$. Then $H - (S_0 + x - s_m)$ has fewer B_{h-1}-components than $H - S_0$, and $S_0 + x - s_m$ is a superstable set. This contradicts the assumption that S_0 miminizes the number of B_{h-1}-components of $H - S_0$.

Thus, whether $h = 3$ or $h > 3$, when P is assumed to exist, we obtain contradictions, and so the theorem is proved.

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